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# Solitons for the cubic–quintic nonlinear Schrödinger equation with time- and space-modulated coefficients

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## Abstract

In this paper, we construct, by means of similarity transformations, explicit solutions to the cubic–quintic nonlinear Schrödinger equation with potentials and nonlinearities depending on both time and spatial coordinates. We present the general approach and use it to calculate bright and dark soliton solutions for nonlinearities and potentials of physical interest in applications to Bose–Einstein condensates and nonlinear optics.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

During the past several years, there have been a great many theoretical and experimental investigations in models based on the nonlinear Schrödinger equation (NLSE). The physical models of this type emerge in various nonlinear physical phenomena, such as nonlinear optics [1, 2], Bose–Einstein condensates [3], biomolecular dynamics [4], and others [5, 6].

One of the simplest extensions of the cubic NLSE is the so-called cubic–quintic nonlinear Schrödinger (CQNLS) model, which, in normalized units and 1D, is

$$i \frac{\partial \psi}{\partial t} = - \frac{\partial^2 \psi}{\partial x^2} + G_1 |\psi|^2 \psi + G_2 |\psi|^4 \psi. \quad (1)$$

The CQNLS equation is another universal mathematical model describing many situations of physical interest and approximating other more complicated ones. It arises in plasma physics [7], condensed matter physics [8], nuclear physics [9], Bose–Einstein condensates [10], nonlinear optics [11], etc. In the case of Bose–Einstein condensates, the cubic and

quintic nonlinearity terms appear as a consequence of the two- and three-body interactions, respectively. Efimov resonances, which are responsible for three-body interactions, have been observed in an ultra-cold gas of cesium atoms [12]. In optics, the CQNLS equation can describe the propagation of an electromagnetic wave in photorefractive materials as long as variable  $t$  represents the propagation coordinate of the wave. The cubic–quintic nonlinearity occurs due to an intrinsic nonlinear resonance in the material, which also gives rise to strong two-photon absorption [13].

In the past decade, techniques for managing nonlinearity [14] have attracted considerable attention. For instance, nonlinearity management arises in optics for transverse beam propagation in layered optical media [15] as well as in atomic physics for the Feshbach resonance of the scattering length of interatomic interactions in BECs [16]. In these situations, one has to deal with the governing equations with the nonlinearity coefficients being functions of time [16, 17] or, equivalently, the variable representing the propagation distance [14, 18–20]. In a recent reference [21], by using similarity transformations, the authors went beyond previous studies considering space- and time-dependent nonlinearities, and constructed explicit solutions of the cubic nonlinear Schrödinger equation with coefficients depending on time and on the spatial coordinates, which are experimentally feasible due to the flexible and precise control of the scattering length achievable in quasi-one-dimensional BECs with tunable interactions.

The aim of this paper is to study explicit solutions of the cubic–quintic nonlinear Schrödinger equation with time- and space-dependent potentials and nonlinearities, which can be written in the following dimensionless form:

$$iu_t = -u_{xx} + V(t, x)u + g_1(t, x)|u|^2u + g_2(t, x)|u|^4u. \quad (2)$$

In the case of Bose–Einstein condensates,  $u(t, x)$  represents the macroscopic wavefunction,  $V(t, x)$  is an external potential,  $g_1(t, x)$  and  $g_2(t, x)$  are the cubic and quintic nonlinear coefficients, corresponding to the two-body and three-body interactions, respectively. The signs of  $g_1(t, x)$  and  $g_2(t, x)$  can be positive or negative, indicating that the interactions are repulsive or attractive, respectively. Thus, in this paper, we construct different types of explicit solutions such as bright and dark soliton solutions. In order to do this, we resort to the similarity transformation technique and analyze a general class of potentials  $V(t, x)$  and nonlinearity functions  $g_1(t, x)$  and  $g_2(t, x)$ .

The paper is organized as follows. In section 2, we introduce the general approach of similarity transformations. In sections 3 and 4, we use the method for constructing explicit solutions of the time-dependent cubic–quintic nonlinear Schrödinger equation. Section 5 contains a remark on the stability of the previous solutions and the conclusions of our results.

## 2. General theory

We consider the cubic–quintic nonlinear Schrödinger equation with time- and space-dependent coefficients  $g_1(t, x)$ ,  $g_2(t, x)$ ,  $V(t, x)$ , equation (2).

Our first goal is to transform equation (2) into the stationary CQNLS equation

$$EU = -U_{XX} + G_1 |U|^2 U + G_2 |U|^4 U, \quad (3)$$

where both  $U \equiv U(X)$  and  $X \equiv X(t, x)$  are real functions,  $E$  denotes the eigenvalue of the nonlinear equation (which corresponds to the chemical potential in the Bose–Einstein condensate framework and the propagation constant in nonlinear optics), and  $G_1$  and  $G_2$  are the (constant) nonlinearity parameters.

To connect solutions of equation (2) with those of equation (3) we will use the transformation

$$u(t, x) = r(t, x)U [X(t, x)], \tag{4}$$

requiring  $U(X)$  to satisfy equation (3) and  $u(t, x)$  to be a solution of equation (2). The potential  $V(t, x)$  and the nonlinearities  $g_1(t, x)$  and  $g_2(t, x)$  are determined after the transformation is applied. Although the methodology of this transformation has been explained in [21] for the cubic nonlinear Schrödinger equation, for the sake of completeness we show here the method. Moreover, the CQNLSE presents an extra term with respect to the CNLSE, which implies extra complications in the treatment of the solutions. Other similarity transformations have been studied for NLS equations in different contexts [22, 23].

In our calculations, it is convenient to introduce the polar form for the complex modulating function  $r = r(t, x)$

$$r(t, x) = \rho(t, x) e^{i\varphi(t,x)}, \tag{5}$$

where  $\rho$  and  $\varphi$  are real functions ( $\rho$  is also non-negative). Upon the substitution of equation (4) into equation (2), we impose the following system of conditions on  $g_1, g_2$  and  $V$ :

$$g_1(t, x) = G_1 \frac{X_x^2}{\rho^2}, \tag{6}$$

$$g_2(t, x) = G_2 \frac{X_x^2}{\rho^4}, \tag{7}$$

$$V(t, x) = \frac{\rho_{xx}}{\rho} - \varphi_t - \varphi_x^2 - EX_x^2, \tag{8}$$

together with

$$\begin{cases} \rho\rho_t + (\rho^2\varphi_x)_x = 0, \\ (\rho^2X_x)_x = 0, \\ X_t + 2\varphi_x X_x = 0, \end{cases} \tag{9}$$

in order to arrive at a set of non-trivial solutions to equation (2). By solving (9) one gets

$$X_x(t, x) = \frac{\gamma^2(t)}{\rho^2(t, x)} \tag{10}$$

( $\gamma(t)$  is an arbitrary positive definite function of time) and

$$X_{tx}X_x - X_tX_{xx} = \frac{\gamma_t}{\gamma} X_x^2. \tag{11}$$

This last equation yields

$$\left(\frac{X_t}{X_x}\right)_x = \frac{\gamma_t}{\gamma}, \tag{12}$$

which, after integration, results in

$$X(t, x) = F(\xi), \quad \xi(t, x) = \gamma(t)x + \delta(t), \tag{13}$$

where both  $\gamma(t)$  and  $\delta(t)$  are differentiable functions.

Thus, we have proven the following result: the substitution

$$u(t, x) = \rho(t, x) e^{i\varphi(t,x)}U[X(t, x)], \tag{14}$$

where

$$\rho(t, x) = \sqrt{\frac{\gamma}{F'(\xi)}}, \tag{15}$$

$$\varphi(t, x) = -\frac{\gamma_t}{4\gamma}x^2 - \frac{\delta_t}{2\gamma}x + \varepsilon, \tag{16}$$

and  $\varepsilon(t)$  is a differentiable arbitrary function, leads to equation (2), with

$$g_1(t, x) = G_1 \frac{\gamma^4}{\rho^6}, \tag{17}$$

$$g_2(t, x) = G_2 \frac{\gamma^4}{\rho^8}, \tag{18}$$

$$V(t, x) = \frac{\rho_{xx}}{\rho} - \varphi_t - \varphi_x^2 - E \frac{\gamma^4}{\rho^4}, \tag{19}$$

where the function  $\rho(x, t)$  must be sign definite and  $C^2$  (i.e. twice continuously differentiable) for  $g_1(t, x)$ ,  $g_2(t, x)$  and  $V(t, x)$  to be properly defined. Therefore, choosing  $\delta(t)$ ,  $\gamma(t)$  and  $F(\xi)$  (or equivalently  $\rho(x, t)$ ) we can generate  $g_1(t, x)$ ,  $g_2(t, x)$  and  $V(t, x)$  for which the solutions of equation (2) can be obtained from those of equation (3) via equations (4). We will exploit this fact to construct soliton solutions exhibiting interesting non-trivial behavior.

### 3. Exact solutions I

In this section, we first address solutions to equation (2) when the potential  $V(t, x)$  depends on both time and space while the nonlinearities are of the form

$$g_1(t, x) = G_1 \gamma(t), \tag{20}$$

$$g_2(t, x) = G_2. \tag{21}$$

These nonlinearities can be derived by taking

$$\rho(t, x) = \sqrt{\gamma(t)}, \tag{22}$$

which corresponds to the choice  $F(\xi) = \xi$ . From equation (19), one finds that the potential can be cast as

$$V(t, x) = f(t)x^2 + h(t)x - \left( \varepsilon_t + \frac{\delta_t^2}{4\gamma} + E\gamma^2 \right), \tag{23}$$

where

$$f(t) = \frac{\gamma_{tt}\gamma - 2\gamma_t^2}{4\gamma^2}, \tag{24}$$

$$h(t) = \frac{\delta_{tt}\gamma - 2\delta_t\gamma_t}{2\gamma^2}. \tag{25}$$

Moreover,  $X(t, x)$  becomes

$$X(t, x) = \int_0^\xi \frac{\gamma}{\rho^2} d\xi = \xi = \gamma(t)x + \delta(t). \tag{26}$$

On the other hand, many solutions of equation (3) are known. In this paper, we use the bright soliton, given by

$$U = \frac{\eta}{(\sqrt{1 - \beta} \cosh(2\sqrt{-E}x) + 1)^{1/2}}, \tag{27}$$

where

$$\eta = \sqrt{\frac{4E}{G_1}}, \quad \beta = \frac{-16EG_2}{3G_1^2}. \tag{28}$$

Since we assume that  $G_2 > 0$  and  $E < 0$  we have that  $G_1 < 0$ . Thus, a restriction on  $\beta$  is that  $0 \leq \beta < 1$ . Another solution, representing a dark soliton, is given by the following expression:

$$U = \sqrt{a_1(\operatorname{sech}(\mu X) - 1)}, \tag{29}$$

with

$$\mu^2 = -\frac{4}{5}E \tag{30}$$

and

$$G_1 = \frac{8E}{5a_1}, \quad G_2 = \frac{3E}{5a_1^2}, \tag{31}$$

and  $a_1 < 0$  being an arbitrary constant. Thus  $G_1$  is positive while  $G_2$  is negative.

In what follows, we deal with different cases which turn out to be of physical relevance. It is worth remarking that the solutions which will be shown below cannot be calculated in the pure cubic equation, since in the cubic–quintic NLSE, nonlinearities compete with each other when they have different signs. Thus, the cubic–quintic NLS equation, in some sense, generalizes the cubic case.

### 3.1. Solutions in the free space with constant nonlinearities

We take

$$\gamma(t) = 1, \quad \delta(t) = vt, \tag{32}$$

with  $v \in \mathbb{R}$  and  $\varepsilon = -(v^2/4 + E)t$ . Hence, the external potential and the nonlinearities become

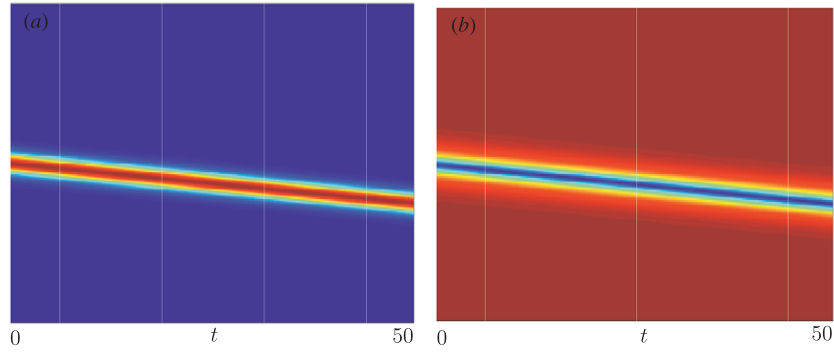
$$V(t, x) = 0, \quad g_1(t, x) = G_1, \quad g_2(t, x) = G_2. \tag{33}$$

The solution is then given by

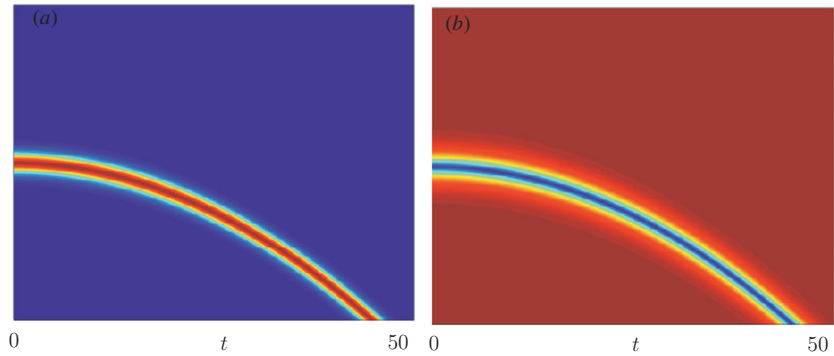
$$u(t, x) = e^{i\varphi(t,x)}U[\xi(t, x)], \tag{34}$$

with  $\varphi(t, x) = -(v/2)x - (v^2/4 + E)t$  and  $\xi(t, x) = x + vt$ , where  $U$  is given by equation (27) in the bright soliton case, and by equation (29) in the dark soliton case. In figures 1(a) and (b), we plot these solutions corresponding to equation (34), for  $v = 0.1$ .

This case, which corresponds to a soliton moving in the free space with a velocity  $v$ , reproduces the Galilean invariance of the system providing consequently a validation of the similarity transformation for the trivial case.



**Figure 1.** (a) Pseudocolor plot of  $|u(t, x)|^2$  where  $u(t, x)$  is the solution of equation (2), for  $V(t, x) = 0$  and  $g_1(t, x) = G_1$ ,  $g_2(t, x) = G_2$ , and for (a) bright soliton profiles ( $E = -1$ ,  $G_1 = -2$ ,  $G_2 = 0.5$ ) and (b) dark soliton profiles ( $E = -1$ ,  $a_1 = -1$ ). In both cases  $x \in [-20, 20]$  and  $\nu = 0.1$ .



**Figure 2.** (a) Pseudocolor plot of  $|u(t, x)|^2$  where  $u(t, x)$  is the solution of equation (2), for potential and nonlinearities given by equations (36), and for (a) bright soliton profiles ( $E = -1$ ,  $G_1 = -2$ ,  $G_2 = 0.5$ ) and (b) dark soliton profiles ( $E = -1$ ,  $a_1 = -1$ ). In both cases  $x \in [-20, 20]$  and  $\nu = 0.01$ .

### 3.2. Solitons with linear potential and constant nonlinearities

If we choose

$$\gamma(t) = 1, \quad \delta(t) = \nu t^2, \tag{35}$$

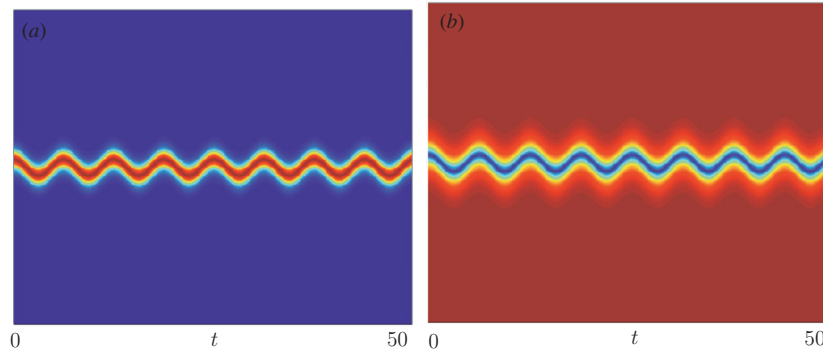
the external potential and the nonlinearities become

$$V(t, x) = \nu x, \quad g_1(t, x) = G_1, \quad g_2(t, x) = G_2. \tag{36}$$

The solution is then given by

$$u(t, x) = e^{i\varphi(t,x)} U[\xi(t, x)], \tag{37}$$

with  $\varphi(t, x) = -\nu t x - \nu^2 t^3 / 3 - E t$  and  $\xi(t, x) = x + \nu t^2$ , where  $U$  is given by equation (27) in the bright soliton case, and by equation (29) in the dark soliton case. In figures 2(a) and (b), we plot these solutions corresponding to equation (37), for  $\nu = 0.01$ .



**Figure 3.** (a) Pseudocolor plot of  $|u(t, x)|^2$  where  $u(t, x)$  is the solution of equation (2), for potential and nonlinearities given by equations (38), and for (a) bright soliton profiles ( $E = -1, G_1 = -2, G_2 = 0.5$ ) and (b) dark soliton profiles ( $E = -1, a_1 = -1$ ). In both cases  $x \in [-20, 20]$ .

### 3.3. Solitons with a linear potential modulated by a time sinusoidal function and constant nonlinearities

Taking  $\gamma(t) = 1$  and  $\delta(t) = -\cos(\Omega t + \beta_0)$ , we obtain

$$v(t, x) = \frac{\Omega^2}{2} \cos(\Omega t + \beta_0)x, \quad g_1(t, x) = G_1, \quad g_2(t, x) = G_2. \quad (38)$$

With this choice, the solution is given by

$$u(t, x) = e^{i\varphi(t,x)} U[\xi(t, x)], \quad (39)$$

where

$$\varphi(t, x) = -\frac{\Omega}{2} \sin(\Omega t + \beta_0)x - \frac{\Omega^2}{8}t + \frac{\Omega}{16} \sin(2\Omega t + 2\beta_0) - Et, \quad (40)$$

and  $U$  is given by equation (27) for the bright soliton, and by equation (29) for the dark soliton. In figures 3(a) and (b), we plot these solutions corresponding to equation (39).

## 4. Exact solutions II

In this section, we show two extra examples among the vast amount of inhomogeneous models. The first one is the case of localized nonlinearities in space and time with no potential. In the second case, we calculate solutions for equation (2) with both periodic potential and nonlinearities. We think that our results can be applied to Bose–Einstein condensates with optically controlled interactions [24].

### 4.1. Localized nonlinearities in space and time

In this subsection, we address to nonlinearities localized in space and time. We focus on the localized nonlinearities

$$g_1(t, x) = G_1 \gamma(t) (\text{sech}(\xi))^3, \quad g_2(t, x) = G_2 (\text{sech}(\xi))^4, \quad (41)$$

being the external potential

$$V(t, x) = \frac{\gamma^2(t)}{4} (1 + (1 - 4E) (\text{sech}(\xi))^2) + f(t)x^2 + h(t)x + m(t), \quad (42)$$



where  $f(t)$  and  $h(t)$  are given by equations (24) and

$$m(t) = - \left( \varepsilon_t + \frac{\delta_t^2}{4\gamma^2(t)} \right). \tag{43}$$

Our choice corresponds to

$$\rho(t, x) = \sqrt{\gamma(t) \cosh(\xi)}. \tag{44}$$

The potential  $V(t, x)$  vanishes by taking  $E = 1/4$ ,  $\gamma(t) = 1$ ,  $\delta(t) = 0$  and  $\varepsilon(t) = t/4$ . A solution of equation (3) with  $E = 1/4$  is given by

$$U = \frac{\operatorname{sn}(\mu X, k)}{\sqrt{a_0 + a_1 \operatorname{dn}^2(\mu X, k)}}, \tag{45}$$

with the requirement  $a_0 > |a_1|$ ,  $k$  being the modulus of the Jacobian elliptic functions

$$\mu = \sqrt{\frac{-E(a_0 + a_1)}{2k^2 a_1 - k^2 a_0 - a_0 - a_1}} \tag{46}$$

and

$$G_1 = \frac{2Ek^2(a_0^2 - 2a_1a_0k^2 - a_1^2 + a_1^2k^2)}{a_0k^2 + a_0 - 2k^2a_1 + a_1}, \tag{47}$$

$$G_2 = \frac{3Ea_1a_0k^4(a_0 + a_1 - k^2a_1)}{k^2a_0 + a_0 - 2k^2a_1 + a_1}. \tag{48}$$

We demonstrate below that equation (2) in the free space with localized nonlinearities given by equation (41) can support bound states with an arbitrary number of solitons, resembling the results of [21].

In this particular case, as

$$X(t, x) = \int_0^\xi \frac{\gamma(t)}{\rho^2(t, x)} d\xi, \tag{49}$$

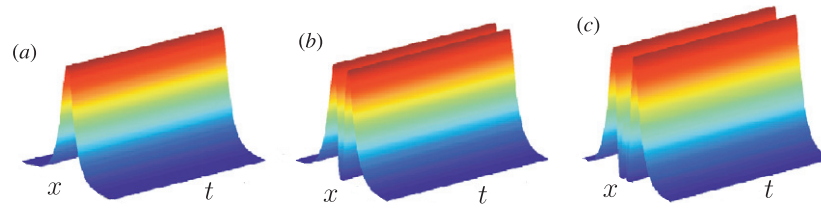
one can obtain  $\cos X = -\tan(\xi)$ , thus  $0 < X < \pi$ . In order to meet the boundary conditions  $\psi(\pm\infty) = 0$ , one has to impose  $U(0) = U(\pi) = 0$ . Evidently,  $U(0) = 0$  is satisfied for the solution (45) and in order to meet  $U(\pi) = 0$ , the condition  $\mu\pi = 2nK(k)$ , where  $K(k)$  is the elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2(x)}}, \tag{50}$$

must hold. Hence for every  $n$ , we can find a value of  $k$  leading to the following family of solutions:

$$\psi(t, x) = \sqrt{\cosh(\xi)} e^{it/4} U(X(t, x)), \tag{51}$$

where  $X(t, x)$  is given by equation (49) and  $U$  is given by equation (45). The solutions corresponding to  $n = 1, 2, 3$  are depicted in figure 4. It is observed that  $\psi$  has exactly  $n - 1$  zeros.



**Figure 4.** Plots of  $|u(t, x)|^2$  for solutions of equation (2), given by equation (51), corresponding to (a)  $n = 1$ , (b)  $n = 2$  and (c)  $n = 3$ . In all cases,  $x \in [-7, 7]$  and  $t \in [0, 50]$ .

4.2. Periodic potential and nonlinearities

Finally, as a last application of our method, we take  $\rho(t, x) = \sqrt{\gamma(t)(1 + \alpha \cos(\omega\xi))}$ . In this way, nonlinearities  $g_1(t, x)$  and  $g_2(t, x)$  are given by

$$g_1(t, x) = G_1 \gamma(t) (1 + \alpha \cos(\omega\xi))^{-3}, \tag{52}$$

$$g_2(t, x) = G_2 (1 + \alpha \cos(\omega\xi))^{-4}. \tag{53}$$

If, moreover, we take  $E = -\alpha\omega/2$ , with  $0 < \alpha < 1$  and  $\omega > 0$ , the external potential is given by

$$V(t, x) = -\frac{\alpha\omega^2\gamma^2}{2} \frac{2 + \alpha \cos(\omega\xi)}{(1 + \alpha \cos(\omega\xi))^2} + f(t)x^2 + h(t)x - \varepsilon_t - \frac{\delta_t^2}{4\gamma}. \tag{54}$$

Taking  $\gamma = 1, \delta = \nu t$ , for  $\nu \in \mathbb{R}$  and  $\varepsilon = -\nu^2 t/4$ , we get the periodic potential

$$V(t, x) = -\frac{\alpha\omega^2}{2} \frac{2 + \alpha \cos(\omega\xi)}{(1 + \alpha \cos(\omega\xi))^2}, \tag{55}$$

and the nonlinearities are finally given by

$$g_1(t, x) = G_1 (1 + \alpha \cos(\omega\xi))^{-3}, \tag{56}$$

$$g_2(t, x) = G_2 (1 + \alpha \cos(\omega\xi))^{-4}, \tag{57}$$

where  $\xi = x + \nu t$ . For small  $\alpha$  these nonlinearities are approximately harmonic

$$g_1(x) \simeq G_1 (1 - 3\alpha \cos(\omega\xi)), \quad g_2(x) \simeq G_2 (1 - 4\alpha \cos(\omega\xi)), \quad \alpha \ll 1. \tag{58}$$

We can construct our canonical transformation by using equation (49) and obtain

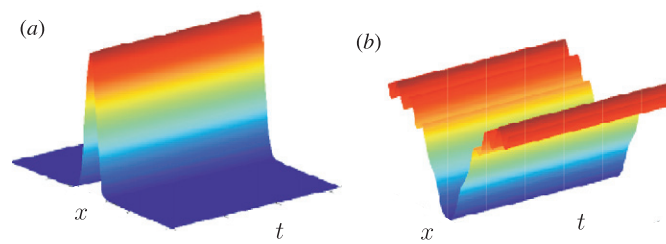
$$\tan\left(\frac{\omega}{2}\sqrt{1 - \alpha^2}X(t, x)\right) = \frac{\sqrt{1 - \alpha}}{1 + \alpha} \tan\left(\frac{\omega\xi}{2}\right). \tag{59}$$

Again, we use equations (27) and (29) as solutions of equation (3). Thus, solutions of equation (2) are given by

$$\psi(t, x) = \sqrt{1 + \alpha \cos(\omega\xi)} e^{i\varphi(t, x)} U(X), \tag{60}$$

where  $\varphi(t, x) = -\nu x/2 + \varepsilon$  and  $U(X)$  is given by equation (27) or (29).

In figure 5, we have drawn different solutions of equation (2) corresponding to bright and dark soliton solutions for  $\nu = 0$ .



**Figure 5.** Plots of  $|u(t, x)|^2$  for different solutions of equation (2), given by equation (60), corresponding to  $\nu = 0$ ,  $\omega = 1$  for (a) the bright soliton solution, with  $\alpha = 0.5$  and (b) the dark soliton solution, with  $\alpha = 0.1$ . In both cases,  $x \in [-15, 15]$  and  $t \in [0, 120]$ .

## 5. Conclusions

In this paper, we have used similarity transformations to find exact solutions of the cubic–quintic nonlinear Schrödinger equation with time- and space-modulated nonlinearities and potentials. We have explicitly calculated bright and dark soliton solutions of equation (2), with nonlinearities and potentials of physical interest in applications to Bose–Einstein condensates and nonlinear optics. The ideas contained in this paper can also be extended to study multicomponent systems, higher-dimensional profiles, etc. We hope that this paper will stimulate further research on these topics and help to understand the behavior of nonlinear waves in systems where not only the potentials, but also the nonlinearities, are inhomogeneous in space and time.

We have also tried to study the stability of these exact solutions by means of a direct simulation, taking as initial condition  $u(0, x)$ . These simulations were performed by means of a finite-difference discretization of the spatial derivatives together with a fifth-order Dormand–Prince integrator (i.e. a Runge–Kutta scheme with stepsize control). Simulations prove the stability of the bright and dark solitons in the free space (34) and with a linear potential (37). However, simulations for time-modulated potentials (that is, (39) and those of section 4) are cumbersome as they need a special choice of boundary conditions. We are currently working on this issue and on the orbital stability of solitons in time periodic potentials. The results of these studies will be the aim of further publications.

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